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Geometry of Schrödinger space-times, global coordinates, and harmonic trapping

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Abstract: We study various geometrical aspects of Schrödinger space-times with dynamical exponent $z > 1$ and compare them with the properties of AdS ($z = 1$). The Schrödinger metrics are singular for $1 < z < 2$ while the usual Poincaré coordinates are incomplete for $z \geq 2$. For $z = 2$ we obtain a global coordinate system and we explain the relations among its geodesic completeness, the choice of global time, and the harmonic trapping of non-relativistic CFTs. For $z > 2$, we show that the Schrödinger space-times admit no global timelike Killing vectors.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence, Space-Time Symmetries

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Contents

1 Introduction

Recently, initiated by [1, 2], there has been some interest in extending the AdS/CFT correspondence to non-relativistic field theories in d spatial dimensions that exhibit an anisotropic scale invariance $(t, x^i) \to (\lambda^z t, \lambda x^i)$ parametrised by the dynamical critical exponent $z \geq 1$, and corresponding to a dispersion relation of the form $\omega \sim k^z$. While there is a plethora of non-relativistic symmetry algebras, some of them are subalgebras of the relativistic conformal (or AdS isometry) algebra. Systems exhibiting such a symmetry therefore potentially have bulk gravitational duals that can be realised as suitable deformations of AdS. The simplest of these are the Lifshitz and Schrödinger space-times Lif_z [3] and Sch_z [1, 2], whose metrics in Poincaré-like coordinates take the form

$$\begin{aligned}
\mathsf{Lif}_z : ds^2 &= -\frac{dt^2}{r^{2z}} + \frac{1}{r^2}\left(dr^2 + d\vec{x}^2\right) \\
\mathsf{Sch}_z : ds^2 &= -\frac{dt^2}{r^{2z}} + \frac{1}{r^2}\left(-2dtd\xi + dr^2 + d\vec{x}^2\right)
\end{aligned} \tag{1.1}$$

where $d\vec{x}^2 = (dx^1)^2 + \ldots (dx^d)^2$. Subsequently, various geometrical aspects of such a non-relativistic correspondence were investigated e.g. in [4]–[16].[1] Nevertheless it is probably fair to say that the holographic dictionary and the issue of holographic renormalisation in these space-times are not yet nearly as well understood as in the AdS case.

In the usual AdS/CFT correspondence, while for most practical intents and purposes it is sufficient to work in Euclidean signature (an option not readily available for the Sch_z metrics) or perhaps on the Minkowskian Poincaré patch, certain conceptual issues of the correspondence are greatly clarified by formulating the Lorentzian correspondence in global

[1] For an updated account of these developments, and references to the CFT side of the story, see also [17].

coordinates (see e.g. [18–20]). For these reasons, and in order to highlight the analogies respectively differences between AdS ($z = 1$) and $z > 1$, it is important to gain a better understanding of the global geometry of the Lifshitz and Schrödinger space-times.

For example, while Lif_z and Sch_z are geodesically complete at $r \to 0$ for all $z \geq 1$, for $z > 1$ the detailed behaviour of geodesics near $r = 0$ differs somewhat from the AdS case ($r = 0$ is "harder to reach"), and this may well have implications for holography and, in particular, for an appropriate notion of "boundary" in this context.

At the "other end" $r \to \infty$, for all $z \geq 1$ the above Poincaré-like coordinate system (1.1) is incomplete in the sense that e.g. timelike geodesics can reach $r = \infty$ in finite proper time. The implications of this run-away behaviour of the geodesics, i.e. whether this indicates a genuine pathology of the space-time (geodesic incompleteness, singularity) or a mere coordinate singularity, requiring one to extend the space-time beyond $r = \infty$, depend on the behaviour of the geometry as $r \to \infty$. For example, it is of course well known that in the $z = 1$ AdS case the above Poincaré coordinates cover only one-half of the complete (non-singular and maximally symmetric) AdS space-time. On the other hand it has already been noted in [3, 17] that for all $z > 1$ the Lifshitz geometries are singular as $r \to \infty$ in the sense of pp-curvature singularites (infinite tidal forces) and are thus geodesically incomplete. For a discussion of the possible implications of this for the Lif_z / CFT correspondence see [17].

The situation is somewhat more interesting for the Schrödinger metrics Sch_z. Our starting point is the observation that in this case qualitatively (for the precise statement see (2.5) below) the tidal forces of causal geodesics behave as

$$\mathsf{Sch}_z : \text{Tidal Forces} \propto (z-1)r^{4-2z} \quad . \tag{1.2}$$

In particular, while these space-times are geodesically incomplete for $1 < z < 2$, there are no infinite tidal forces not only for the AdS case $z = 1$ but also for all $z \geq 2$, and there are freely falling observers that reach $r = \infty$ in finite proper time without encountering any singularity. One thus needs to provide them with a map and extend the space-time beyond $r = \infty$.

In this note we will address this issue and obtain a global, geodesically complete, coordinate system for $z = 2$. We also show that the Schrödinger space-times for $z > 2$ admit no global timelike Killing vector fields, so that a global metric will necessarily be time-dependent.

Taking our clue from global AdS, where global time corresponds to the generator $P_0 + K_0$ of the isometry algebra (K_0 is a special conformal transformation), we oberve that only the $z = 2$ Schrödinger algebra has a potential counterpart of this generator, namely $H + C$, where H is the generator of t-translations (in the above Poincaré-like coordinates) and C is the special conformal generator of the $z = 2$ algebra. By considering the combination $H + \omega^2 C$ we are led to the metric

$$ds^2 = -\frac{dT^2}{R^4} + \frac{1}{R^2}(-2dTdV - \omega^2(R^2 + \vec{X}^2)dT^2 + dR^2 + d\vec{X}^2) \quad . \tag{1.3}$$

which has a number of remarkable properties. First of all, this coordinate system, in which the metric simply has the form of a plane wave deformation of the Poincaré-like metric (1.1), is indeed geodesically complete for $\omega > 0$ and in this sense provides global coordinates for the $\mathsf{Sch}_{z=2}$ space-time (for $\omega = 0$ the metric reduces to the incomplete Poincaré-patch metric (1.1)). Moreover, this metric is closely related to the harmonic trapping of non-relativstic CFTs that plays an important role in the non-relativistic operator-state correspondence [21] and whose holographic implementation was investigated in [4, 5]. Our derivation of the above metric shows that precisely for $z = 2$ (and for AdS $z = 1$) the plane wave deformation (1.3) of the Sch_z Poincaré metric (1.1) that accomplishes this trapping is just a coordinate transformation, namely the one that relates the Poincaré time Hamiltonian $H = \partial_t$ to the trapped Hamiltonian $H+\omega^2 C = \partial_T$. The geodesic completeness of this coordinate system can be physically understood in terms of the trapping of geodesics induced by the harmonic oscillator term $\omega^2(R^2 + \vec{X}^2)$ in the metric. Moreover, the spatial harmonic oscillator provides an IR cut-off that is the counterpart of the topological spatial IR cut-off (space is a sphere) provided by AdS global coordinates.

To set the stage, in section 2 we briefly recall some elementary aspects of the geometry (isometries, geodesics) of the Lif_z and Sch_z metrics in the Poincaré-like coordinates (1.1). In section 3.1, we motivate the introduction of $H + C$ as the generator of global time by analogy with AdS, and we show that the Schrödinger space-times for $z \neq 1, 2$ have no global timelike Killing vectors. We obtain the desired coordinate transformation and the metric in global coordinates in section 3.2, and in section 3.3 we establish the geodesic completeness and discuss the other results mentioned above. In section 3.4 we briefly look at some related issues for pure AdS ($z = 1$) and make some comments on the case $z > 2$. Finally, In section 4, we analyse the Klein-Gordon equation in global coordinates and compare with the Poincaré-patch analysis of [1, 2] and the Hamiltonian analysis of [5].

2 Schrödinger and Lifshitz space-times in Poincaré coordinates

In this section, to motivate our investigation, and as a preparation for the considerations of section 3, we briefly summarise some basic facts about the geometry of the Schrödinger and Lifshitz space-times, whose metrics in Poincaré-like coordinates (that we will henceforth simply refer to as Poincaré coordinates) have been given in (1.1). Obviously for $z = 1$ these reduce to the $(d+2)$- (respectively $(d+3)$-) dimensional AdS Poincaré metric, and we will consider the range $z \geq 1$.

In addition to the manifest translational isometries in t and $\vec{x}$ and spatial rotations these space-times have the characteristic anisotropic dilatation symmetry

$$\begin{aligned} \mathsf{Lif}_z &: (r, \vec{x}, t) \to (\lambda r, \lambda\vec{x}, \lambda^z t) \\ \mathsf{Sch}_z &: (r, \vec{x}, t, \xi) \to (\lambda r, \lambda\vec{x}, \lambda^z t, \lambda^{2-z}\xi) \ . \end{aligned} \tag{2.1}$$

These comprise the so-called Lifshitz algebra [3]. The larger Schrödinger isometry algebra of Sch_z contains, in addition, Galilean boosts and null translations in ξ, the latter playing the role of the central extension or mass operator of the Galilean algebra. Moreover, for $z = 2$ there is one extra special conformal generator C which will turn out to play an important role in the considerations of section 3.

One can use the conserved momenta $E, \vec{P}$ (and P_ξ) corresponding to the manifest t, $\vec{x}$ (and ξ) translational isometries of the metrics (1.1) to reduce the geodesic equations to a single radial (effective potential) equation

$$\begin{aligned} \mathsf{Lif}_z : k &= \frac{\dot{r}^2}{r^2} + r^2\vec{P}^2 - r^{2z}E^2 \\ \mathsf{Sch}_z : k &= \frac{\dot{r}^2}{r^2} + r^2(\vec{P}^2 - 2EP_\xi) + r^{4-2z}P_\xi^2 \end{aligned} \tag{2.2}$$

($k = 0, \mp 1$ for null, timelike and spacelike geodesics). We will first compare and contrast the qualitative behaviour of causal AdS geodesics ($z = 1$) as $r \to 0$ with that for $z > 1$, and then consider the (for our purposes more crucial) behaviour as $r \to \infty$.

In the AdS case $\mathsf{Lif}_{z=1}$ it follows from (2.2) that timelike geodesics require $E^2 - \vec{P}^2 \equiv M^2 > 0$, and that these have a minimal radius $r_{\text{min}} = 1/M$, while null geodesics ($\dot{r} = \pm Mr^2$) can reach $r = 0$ at infinite values of the affine parameter. Since $\dot{t} = Er^2$, it also follows that $r(t) = \pm(E/M)t$, so that lightrays can reach the boundary $r = 0$ and bounce back again to a stationary observer in finite coordinate time t.

The behaviour of $\mathsf{Lif}_{z>1}$ causal geodesics is qualitatively similar to the AdS case, with one perhaps crucial difference: namely, timelike geodesics still have a minimal radius $r_{\text{min}} > 0$, but here so do null geodesics unless $\vec{P} = 0$ (since $r^2\vec{P}^2$ dominates over $r^{2z}E^2$ as $r \to 0$ unless $\vec{P} = 0$). Thus for $z > 1$ only purely radial null geodesics reach $r = 0$, and up to a reparametrisation $r^z \to r$ these are identical to null geodesics in AdS_2.

The $\mathsf{Sch}_{z>1}$ space-times exhibit a somewhat stronger deviation from the AdS behaviour, since here neither timelike nor null geodesics ever reach $r = 0$. This is due to the fact that for $z > 1$ the dominant term in the effective potential is the positive term $r^{4-2z}P_\xi^2$ unless $P_\xi = 0$, and that there are neither timelike geodesics, nor null geodesics with $\dot{r} \neq 0$, for $P_\xi = 0$.

Now let us look at the behaviour as $r \to \infty$. It is easy to see that for all $z \geq 1$ the Poincaré coordinate system (1.1) is incomplete. Indeed, it follows immediately from (2.2) that for $z \geq 1$ the leading large r behaviour of null (and timelike) geodesics as functions of the affine parameter τ is

$$\begin{aligned} \mathsf{Lif}_z &: r(\tau) \propto |\tau - \tau_0|^{-1/z} \\ \mathsf{Sch}_z &: r(\tau) \propto |\tau - \tau_0|^{-1} \quad \forall z \geq 1 \end{aligned} \tag{2.3}$$

so that $r \to \infty$ for $\tau \to \tau_0$. Generically, i.e. unless some of the constants of motion are set to zero, all other coordinates also approach infinity (for Sch_z at exactly the same rate as $r(\tau)$).

In order to assess the implications of this, one needs to look more closely at the geometry of the space-time at $r \to \infty$. The AdS case $z = 1$ is of course well understood: $r = \infty$ is only a coordinate singularity, Poincaré coordinates cover only one-half of the complete AdS space-time, and it is possible to introduce global coordinates that cover the entire space-time. It is also easy to see that (for any z) all scalar curvature invariants are constant (and in particular finite at $r = \infty$). This is a consequence of the homogeneity of the Lifshitz and Schrödinger space-times, in particular the dilatation isometry (2.1), since any scalar curvature invariant can only be a function of r, upon which dilatation-invariance implies that the invariant is actually constant. However, as is well known, e.g. in the context of pp-waves, all of whoses scalar curvature invariants are identically zero, this does not by itself imply that the space-time should necessarily be considered to be non-singular: freely falling observers may nevertheless experience infinite tidal forces (and therefore a physical singularity), in the form of divergent parallel propagated orthonormal frame components of the Riemann tensor.

The explicit calculation of the tidal forces is pretty straightforward in the case of Lifshitz metrics, and it has already been noted in [3] that they are singular in this sense for $r \to \infty$. One finds that for null geodesics (and up to a cosmological constant term for timelike geodesics) the tidal forces are proportional to

$$\mathsf{Lif}_z : \text{Tidal Forces} \propto (z-1)r^{2z} \quad . \tag{2.4}$$

This is in complete agreement with the result reported in [17, fooootnote 5] and shows that the Lif_z space-times are geodesically incomplete and singular at $r \to \infty$ for all $z > 1$.

For the Schrödinger metrics the corresponding calculation is slightly more involved but the result is also somewhat more interesting and qualitatively quite different from (2.4). The relevant parallel propagated orthonormal frame components $R^{(\tau)}_{\ (\alpha)(\tau)(\beta)}$ of the curvature tensor are

$$\mathsf{Sch}_z : \text{Tidal Forces:} \left\{ \begin{array}{ll} R^{(\tau)}_{\ (i)(\tau)(j)} & = -(1+P_\xi^2(z-1)r(\tau)^{4-2z})\delta_{ij} \\ R^{(\tau)}_{\ (\xi)(\tau)(\xi)} & = -(1+2P_\xi^2 z(z-1)r(\tau)^{4-2z}\sin(\tau)^2) \\ R^{(\tau)}_{\ (r)(\tau)(r)} & = -(1+2P_\xi^2 z(z-1)r(\tau)^{4-2z}\cos(\tau)^2) \end{array} \right\} \propto (z-1)r^{4-2z} \tag{2.5}$$

with (τ) referring to the tangent of the timelike geodesic, i.e. $e^\alpha_{(\tau)} = \dot{x}^\alpha$. While, as expected, $z = 1$ is non-singular, this result, that can also be deduced from the calculation of geodesic deviation in general Siklos space-times in [22] and [23], shows some perhaps surprising features.

Namely, while a static (non-geodesic) observer may have have been inclined to believe that the metric is asymptotically AdS for any $z > 1$ as $r \to \infty$, since the $r^{-2z}dt^2$ term appears to be subleading, this is an illusion caused by that observer's acceleration. Indeed (2.5) shows that there is a singularity in the form of infinite tidal forces at $r = \infty$ for $1 < z < 2$, experienced by all timelike and null geodesics (since for these $P_\xi \neq 0$), and the situation is therefore very much like that of the Lifshitz space-times for $z > 1$. For $z \geq 2$, however, the tidal forces again remain finite as $r \to \infty$.[2] *Mutatis mutandis* the above result is also valid for the tidal forces experienced by null geodesics; the first (cosmological constant) term does not contribute in that case. Thus for $z \geq 2$ causal geodesics reach $r = \infty$ at finite values of the affine parameter without encountering any singularity.

[2] For $z > 2$, the dangerous region may appear to be $r \to 0$, but this is deceptive since (as discussed above) timelike and null geodesics have a non-zero minimal radius $r_{\min} > 0$.

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3 Global coordinates for $z = 2$ Schrödinger space-times

The above analysis points to the necessity of constructing suitable coordinates that cover the space-time region beyond $r = \infty$. In this section we will obtain a global, geodesically complete, coordinate system for $\mathsf{Sch}_{z=2}$ and describe some of its properties. We also make some comments on the (qualitatively quite different) case $z > 2$.

3.1 Towards global coordinates

We begin by recalling the situation for AdS, i.e. $\mathsf{Sch}_{z=1}$. In this case, it is well known how to construct global coordinates (T, R, angles) in terms of which the $(d+3)$-dimensional AdS metric takes the form

$$\mathsf{AdS} : ds^2 = -(1+R^2)dT^2 + (1+R^2)^{-1}dR^2 + R^2 d\Omega^2_{d+1} \ . \tag{3.1}$$

The most straightforward way to find these global coordinates is to make use of the embedding of the unit curvature radius $(d+3)$-dimensional AdS space-time into $\mathbb{R}^{2,d+2}$ with coordinates $Z^A, A = 0, \ldots, d+3$ and metric

$$ds^2 = -(dZ^0)^2 + (dZ^1)^2 + \ldots + (dZ^{d+2})^2 - (dZ^{d+3})^2 \tag{3.2}$$

as the (universal covering space of the) hyperboloid

$$-(Z^0)^2 + (Z^1)^2 + \ldots + (Z^{d+2})^2 - (Z^{d+3})^2 = -1 \ . \tag{3.3}$$

Writing this as

$$(Z^0)^2 + (Z^{d+3})^2 = 1 + (Z^1)^2 + \ldots + (Z^{d+2})^2 \equiv 1 + R^2 \tag{3.4}$$

suggests the parametrisation

$$Z^0 = (1+R^2)^{1/2}\sin T \qquad Z^{d+3} = (1+R^2)^{1/2}\cos T \ , \tag{3.5}$$

which identifies ∂_T with the generator of rotations $M_{0,d+3}$ in the timelike (Z^0, Z^{d+3})-plane. This indeed gives rise on the nose to the global metric (3.1).

Since the Sch_z metric (1.1) differs from the AdS Poincaré metric only by the characteristic first term $-dt^2/r^{2z}$, when seeking global coordinates for Sch_z, one's first thought may perhaps be to simply employ the usual transformation from AdS Poincaré coordinates

$$ds^2 = \frac{-dt^2 + d\vec{y}^2 + dr^2}{r^2} \ , \tag{3.6}$$

to global coordinates. However, since e.g. the relation between global and Poincaré time is

$$\tan T = \frac{2t}{1 + r^2 + \vec{y}^2 - t^2} \ , \tag{3.7}$$

this results in a fairly complicated metric that explicitly depends on all of the coordinates. In particular, none of the Schrödinger isometries of the metric will be manifest and, regardless of whether or not this procedure leads to a geodesically complete coordinate system for $z \geq 2$, it appears to provide no additional insight into the geometry of Schrödinger space-times.

Another possibility is to try to find a deformation of the AdS embedding (3.3) that breaks the conformal algebra down to its Schrödinger sub-algebra. Unfortunately, it is not hard to see that such an embedding of Sch_z into one dimension higher does not exist: any hypersurface invariant under the Schrödinger algebra turns out to be automatically invariant under the entire conformal algebra, and leads to the standard AdS hyperboloid.

However, there is yet another aspect of the AdS construction that does turn out to generalise to the Schrödinger case, and does provide global coordinates, but only for $z = 2$. Namely, under the usual identification of the generators M_{AB} of $\mathfrak{so}(d+2,2)$ with the generators $(P_\mu, K_\mu, M_{\mu\nu}, D)$ of the relativistic conformal algebra $\mathfrak{conf}(d+1,1)$, such that e.g.

$$P_0 = \partial_t \quad , \quad K_0 = t(r\partial_r + y^i\partial_{y^i}) + \frac{1}{2}(t^2 + r^2 + \vec{y}^2)\partial_t \tag{3.8}$$

in standard Poincaré coordinates (3.6), the definition of AdS global time is equivalent to the identification

$$\partial_T = P_0 + K_0 \ . \tag{3.9}$$

Thus global AdS time "diagonalises" the modified Hamiltonian operator $P_0 + K_0$. In the Schrödinger algebra, the role of the Hamiltonian is played by the lightcone Hamiltonian $H \equiv P_+$, and the Poincaré coordinates (1.1) are such that this Hamiltonian is diagonalised, $H = \partial_t$. Now, generically the Schrödinger algebra does not possess any counterpart of the special conformal generators K_μ. Precisely for $z = 2$, however (for which the Schrödinger algebra can be characterised as the subalgebra of $\mathfrak{conf}(d+1,1)$ that commutes with the lightcone momentum P_-), there is one extra special conformal generator, namely $C \equiv K_-$. In Poincaré coordinates C takes the form

$$C = t(t\partial_t + r\partial_r + x^i\partial_{x^i}) + \frac{1}{2}(r^2 + \vec{x}^2)\partial_\xi \tag{3.10}$$

with

$$[H, C] = D \qquad [D, C] = 2C \qquad [D, H] = -2H \ , \tag{3.11}$$

where

$$D = 2t\partial_t + r\partial_r + x^i\partial_{x^i} \tag{3.12}$$

is the generator of dilatations (2.1) for $z = 2$. Thus for $z = 2$, there is a natural candidate counterpart of the AdS global Hamiltonian $P_0 + K_0$, namely

$$P_+ + K_- = H + C \ . \tag{3.13}$$

As a first check on this we can calculate the norm of this Killing vector in the Poincaré-coordinates of (1.1),

$$||H + C||^2 = -1 - \frac{\vec{x}^2}{r^2} - \frac{(1+t^2)^2}{r^4} \leq -1 \ . \tag{3.14}$$

Here the constant term -1 arises from the cross-term between ∂_t and the r^2-term in C (3.10). Thus unlike $H = \partial_t$, whose norm goes to zero as $r \to \infty$, this Killing vector is

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everywhere timelike in the Poincaré patch and thus has a chance of providing a well-defined notion of time also beyond the Poincaré patch. We will show below that diagonalising this generator of the isometry algebra indeed leads to global time (and other global coordinates) for $z = 2$.

Before turning to that, let us briefly look at the situation for $z \neq 1, 2$. In that case, C is absent but one could e.g. consider a linear combination $H + aD + bP_-$, $P_- = \partial_\xi$, of the Killing vectors that are invariant under spatial rotations. Such Killing vectors necessarily become spacelike somewhere inside the Poincaré patch if $a \neq 0$, while the norm of $H + bP_-$ still goes to zero as $r \to \infty$ (in any case, replacing H by $H + bP_-$ just amounts to passing from (t, ξ) to some linear combinations of t and ξ). Including the remaining Killing vectors (rotations, translations, boosts) in this analysis does not improve the situation. We can therefore conclude that, unlike for $z = 2$, the Schrödinger space-times Sch_z for $z > 2$ have no global timelike Killing vector fields. We will come back to this result in section 3.4.

3.2 Global coordinates for $z = 2$

Since the $z = 2$ algebra has the central element $P_- = \partial_\xi$, we seek new coordinates

$$(t, r, \vec{x}, \xi) \mapsto (T, R, \vec{X}, V) \tag{3.15}$$

in which $H + C$ (3.13) and P_- are simultaneously diagonal,

$$H + C = \partial_T \quad , \quad P_- = \partial_V \quad . \tag{3.16}$$

This is accomplished by the coordinate transformation

$$\begin{aligned} t &= \tan T \quad , \quad r = \frac{R}{\cos T} \quad , \quad \vec{x} = \frac{\vec{X}}{\cos T} \\ \xi &= V + \frac{1}{2}\left(R^2 + \vec{X}^2\right)\tan T \end{aligned} \tag{3.17}$$

(chosen to also keep the metric as diagonal as possible - no off-diagonal terms in the new radial coordinate R - see also section 3.4 for further comments on this transformation), and in these coordinates the metric reads

$$\begin{aligned} \mathsf{Sch}_{z=2}: \; ds^2 &= -\left(\frac{1}{R^4} + \left(1 + \frac{\vec{X}^2}{R^2}\right)\right) dT^2 + \frac{1}{R^2}\left(-2dTdV + dR^2 + d\vec{X}^2\right) \\ &= -\frac{dT^2}{R^4} + \frac{1}{R^2}\left(-2dTdV - (R^2 + \vec{X}^2)dT^2 + dR^2 + d\vec{X}^2\right) \quad . \end{aligned} \tag{3.18}$$

This metric has several noteworthy properties. First of all, it is indeed geodesically complete, i.e. all geodesics can be extended to infinite values of their affine parameter. We postpone a detailed proof of this assertion to section 3.3, but already here draw attention to the fact that a crucial role in establishing this is played by the harmonic oscillator potential $R^2 + \vec{X}^2$ induced by the isotropic plane wave metric $-2dTdV - (R^2 + \vec{X}^2)dT^2 + dR^2 + d\vec{X}^2$ in (3.18) which replaces the flat Poincaré coordinate metric $-2dtd\xi + dr^2 + d\vec{x}^2$. In section 3.3 we will also discuss other issues related to this term and its interpretation in terms of the harmonic trapping [21] of non-relativistic conformal field theories. It may also be useful to note that, as in [7], the above form (3.18) of the $z = 2$ metric can also be obtained from pure AdS (in plane wave coordinates (3.32)) by a TST-transformation.[3]

It is perhaps quite surprising that the above transformation between Poincaré and global cooordinates is so much simpler than its AdS counterpart. For instance, instead of the AdS relation (3.7) one has the simple relation $t = \tan T$ (3.17) between Poincaré and global time for $z = 2$. The success of the simple coordinate transformation (3.19) can be traced back or attributed to the fact (we will briefly recall in section 3.4) that it is precisely the coordinate transformation that exhibits the conformal flatness of isotropic plane waves.

One remarkable (and related) feature of the global metric (3.18) is that it differs from the Poincaré-metric (1.1) only by a single term, namely the plane wave harmonic oscillator frequency term $\sim (dT)^2$. This is in marked contrast to the global AdS metric (3.1) which appears to bear no resemblance whatsoever to the Poincaré metric. This statement can even be sharpened somewhat by introducing a real (and without loss of generality positive) parameter ω into the coordinate transformation (3.17), via

$$\begin{aligned} &t = \omega^{-1} \tan \omega T \quad , \quad r = \frac{R}{\cos \omega T} \quad , \quad \vec{x} = \frac{\vec{X}}{\cos \omega T} \\ &\xi = V + \frac{\omega}{2} \left(R^2 + \vec{X}^2 \right) \tan \omega T \ . \end{aligned} \tag{3.19}$$

In terms of these coordinates the metric now takes the form

$$\mathsf{Sch}_{z=2}: \ ds^2 = -\frac{dT^2}{R^4} + \frac{1}{R^2} \left(-2dTdV - \omega^2 (R^2 + \vec{X}^2) dT^2 + dR^2 + d\vec{X}^2 \right) \ . \tag{3.20}$$

Thus this metric interpolates between the Poincaré metric for $\omega = 0$ (for which (3.19) obligingly reduces to the identity transformation) and the global metric for $\omega = 1$. The metric is actually geodesically complete for any $\omega > 0$ since (3.20) can be obtained from (3.18) by the scaling $(R, T, \vec{X}, V) \to (\sqrt{\omega} R, \omega T, \sqrt{\omega} \vec{X}, V)$. This happens to look very much like the $z = 2$ dilatation symmetry (2.1) in Poincaré coordinates, but acting on global coordinates this is not an isometry but rather the transformation that turns (3.18) into (3.20).

To better understand what is going on here, note that the coordinate transformation (3.19) diagonalises not $H + C$ but $H + \omega^2 C$, so that it is not too surprising that one finds the Poincaré metric for $\omega = 0$ and the global metric (3.18) for $\omega = 1$. Thus any non-trivial linear combination of H and C (with a relative positive coefficient) gives rise to a global Hamiltonian.

While this explains the form (3.20) of the $z = 2$ global metric, this begs the question if one cannot adopt a similar procedure in the AdS case, diagonalising not $P_0 + K_0$ (which, as we know, gives rise to global coordinates) but $P_0 + \lambda^2 K_0$ and finding a metric depending on λ that interpolates between the Poincaré metric for $\lambda = 0$ and the global metric for $\lambda = 1$. At first this may perhaps appear unlikely precisely because the global metric (3.1) is so unlike the Poincaré metric, but nevertheless this is indeed possible. For notational

[3] We are grateful to a referee for stressing this to us.

simplicity, we exhibit this 1-parameter family of interpolating metrics only in the $(3+1)$-dimensional $(d=1)$ case,

$$\mathsf{AdS}: \; ds^2 = -(\lambda^2+R^2)dT^2 + (\lambda^2+R^2)^{-1}dR^2 + R^2\left(d\Theta^2 + \cos^2\left(\frac{\lambda^2\pi}{2} - \lambda\Theta\right)d\Phi^2\right) \; . \tag{3.21}$$

The explicit coordinate transformation, which we will not give here in detail (instead of (3.7) one now has

$$\tan\lambda T = \frac{2\lambda t}{1+\lambda^2(r^2+\vec{y}^2-t^2)} \tag{3.22}$$

etc.) shows that $(\lambda T, \lambda\Theta, \lambda\Phi)$ are standard angles. Thus, on the one hand the above metric reduces to the global metric (3.1) for $\lambda = 1$, while on the other hand for $\lambda \to 0$ the time-coordinate becomes non-compact and the spatial sphere decompactifies to Euclidean space, so that one obtains the standard Poincaré metric (3.6) with $R = 1/r$ and $\{y^i\} = \{\Theta, \Phi\}$.

3.3 The global metric: harmonic trapping and geodesic completeness

There is one aspect of the global $\mathsf{Sch}_{z=2}$ metric constructed above that merits particular attention, and that we already alluded to above, namely its relation to the harmonic trapping of non-relativistic CFTs [21] and its geometric realisation [4, 5]. Recall that we were led to the metric (3.18) by analogy with the AdS case and by the realisation that there is an essentially unique counterpart of the global AdS Hamiltonian $P_0 + K_0$ in the $z=2$ Schrödinger algebra, namely the generator $P_+ + K_- \equiv H + C$.

Non-relativistic CFT, on the other hand, provides an a priori completely different rationale for studying the modified Hamiltonian $H \to H + C$, because the non-relativistic operator-state correspondence [21] relates primary operators of the Schrödinger algebra (those that commute with C and Galilean boosts) with energy eigenstates of $H + C$. Since essentially the effect of C is to add a harmonic potential to the Hamiltonian, this corresponds to putting the system into a harmonic trap.

In [5], the question was investigated how this trapping could be realised holographically via a deformation of the (Poincaré patch) Schrödinger metric (1.1). The deformation that was found to accomplish a harmonic trapping both in the spatial directions of the CFT and in the holographic radial coordinate r turns out (when specialised to $z=2$)[4] to agree precisely with the global metric (3.20). Our derivation of this metric shows that precisely for $z=2$ (and for $z=1$, see section 3.4) the required deformation of the metric that accomplishes this trapping is actually a pure gauge deformation, namely a coordinate transformation that relates the Poincaré time generator $H = \partial_t$ to the trapping (or global) time generator $H + \omega^2 C = \partial_T$.

The plane wave deformation (3.20) of the Poincaré metric (1.1) achieves this trapping deformation of the Hamiltonian for the same reason that the massless Klein-Gordon equation for a scalar field Φ in a pp-wave metric background

$$ds^2 = -2dtd\xi - U(\vec{x},t)dt^2 + d\vec{x}^2 \tag{3.23}$$

[4] The emphasis in [5] (and also in [4]) was on $z=1$ and the attempt to find a pure AdS DLCQ dual realisation of systems with $z=2$ Schrödinger symmetry.

reduces to the Schrödinger equation with a potential $V = mU/2$ in a sector with fixed lightcone momentum = mass,

$$\Box\Phi = 0\ ,\ i\partial_\xi\Phi = m\Phi \quad \Rightarrow \quad i\partial_t\Phi = -\frac{1}{2m}\Delta\Phi + \frac{m}{2}U\Phi\ . \tag{3.24}$$

For an isotropic harmonic oscillator potential, this is precisely the plane wave metric that appears in (3.20), and we will also encounter this trapping of the scalar field in the analysis of the Klein-Gordon equation in the metric (3.20) in the next section.

We will now show that the completeness of the coordinate system (3.19) is a consequence of the harmonic trapping of geodesics induced by this coordinate transformation. To study the effect of the harmonic oscillator term $\omega^2(R^2+\vec{X}^2)$ in the metric on the behaviour of geodesics, let us compare the $z = 2$ Poincaré radial effective potential equation (2.2)

$$k = \frac{\dot{r}^2}{r^2} + r^2(\vec{P}^2 - 2EP_\xi) + P_\xi^2 \tag{3.25}$$

with the corresponding equation one obtains from the global metric (3.20), namely

$$k = \frac{\dot{R}^2}{R^2} + R^2(P^2 - 2EP_V) + P_V^2 + \omega^2 P_V^2 R^4\ . \tag{3.26}$$

A minor difference between (3.25) and (3.26) is the fact that the constant of motion denoted by P^2 in (3.26) arises not like the $\vec{P}^2$-term in (3.25) as a consequence of translation invariance (which (3.20) does not manifest), but rather as the conserved energy

$$P^2 \equiv \left(\frac{1}{R^2}\frac{d}{d\tau}\vec{X}\right)^2 + \omega^2 P_V^2 \vec{X}^2 \tag{3.27}$$

associated to the transverse harmonic oscillator equations

$$\frac{1}{R^2}\frac{d}{d\tau}\left(\frac{1}{R^2}\frac{d}{d\tau}\vec{X}\right) = -\omega^2 P_V^2 \vec{X}\ . \tag{3.28}$$

The main (and crucial) difference between (3.25) and (3.26), however, lies in the last term $\omega^2 P_V^2 R^4$ in (3.26). This term is negligible for $R \to 0$, where (3.26) reduces to (3.25) which, as we already discussed, is well-behaved as $r \to 0$. On the other hand, since this term dominates the large R behaviour, it prevents any geodesic (for any k) from reaching $R = \infty$ (even for infinite values of the affine parameter τ) unless $\omega = 0$ (the Poincaré-patch metric, which as we know is incomplete at large radius) or $P_V = 0$. When $P_V = 0$, the right-hand-side of (3.26) is a sum of squares, and thus only spacelike geodesics $k = +1$ are possible. When $P^2 \neq 0$, there is again a maximal radius, $R_{\max} = 1/P$, and when $P_V = P = 0$, one has $\dot{R} = \pm R$, and thus these are the only geodesics that can reach $R = \infty$, but they only do so for $|\tau| \to \infty$. The motion in the $\vec{X}$-direction is bounded by the harmonic oscillator potential, and that in the remaining (T, V)-directions is determined by that of R and $\vec{X}$ and remains at finite values of the coordinates for all finite τ. This establishes that, as claimed in section 3.2, the $\mathsf{Sch}_{z=2}$ metric written in the coordinates (3.18), (3.20) is geodesically complete.

We close this section with one more remark on the significance of the trapping exhibited by the global metric and the comparison with the global AdS metric (3.1). As is well known the slices of constant R there have the topology $\mathbb{R} \times S^{d+1}$.

$$\begin{aligned} \mathsf{AdS} : ds^2|_{R=const} &= -(1+R^2)dT^2 + R^2 d\Omega^2_{d+1} \\ &\stackrel{R\to\infty}{\longrightarrow} R^2(-dT^2 + d\Omega^2_{d+1}) \end{aligned} \tag{3.29}$$

Thus the spatial part of the induced boundary metric has topology S^{d+1}, with finite volume, and thus in particular provides a topological IR cut-off for the boundary theory. Without committing ourselves to a particular notion of boundary in the Schrödinger case, roughly speaking the dual CFT should be considered to live on the slices of constant R (the holographic coordinate) and V (dual to the particle number). The metric induced on these slices can, via some constant rescalings of the coordinates, be written as

$$\begin{aligned} \mathsf{Sch}_{z=2}:\ ds^2|_{R,V=const.} &\sim -(1+\omega^2|\vec{X}|)^2 dT^2 + d\vec{X}^2 \\ &= -(1+\omega^2\rho^2)dT^2 + d\rho^2 + \rho^2 d\Omega^2_{d-1}\ . \end{aligned} \tag{3.30}$$

Thus in the Schrödinger case there is no topological cut-off, but the trapping in the spatial directions $\vec{X}$ can be thought of as providing an IR cut-off through the harmonic potential. In particular, the induced metric (3.30) has the standard form of the Newtonian limit of a relativistic metric, here in a spherically symmetric gravitational harmonic oscillator potential $\frac{1}{2}\omega^2\rho^2$. This Newtonian limit aspect of the metric of course fits in well with the non-relativistic symmetries and potential dual dynamics.

3.4 Some comments on $z = 1$ and $z > 2$

While we have motivated the coordinate transformation (3.19) through the special (conformal) symmetries that the $z = 2$ Schrödinger algebra and metric possess, we can apply it to the Sch_z metric for any z. If one does that, one finds the metric

$$\mathsf{Sch}_z : ds^2 = -(\cos^2\omega T)^{z-2}\frac{dT^2}{R^{2z}} + \frac{1}{R^2}\left(-2dTdV - \omega^2(R^2+\vec{X}^2)dT^2 + dR^2 + d\vec{X}^2\right)\ . \tag{3.31}$$

Note that under this transformation the r^{-2z} and r^{-2} terms of the Poincaré metric (1.1) do not mix and transform separately into the corresponding terms in the metric (3.31).

Evidently this reduces to (3.20) for $z = 2$. The Sch_z Poincaré metric reduces to the AdS metric either for $z = 1$ or if we set the coefficient of the r^{-2z}-term to zero. Thus by the same token, choosing $z = 1$ or setting the coefficient of the first term to zero in (3.31) (the two choices are related by a simple shift of V), we obtain a 1-parameter family of AdS metrics, namely

$$\mathsf{AdS}:\ ds^2 = \frac{1}{R^2}\left(-2dTdV - \omega^2(R^2+\vec{X}^2)dT^2 + dR^2 + d\vec{X}^2\right)\ . \tag{3.32}$$

This is AdS in trapping coordinates, and that this AdS plane wave is indeed just pure AdS in disguise was already noted in [23][5] and [7] (where it was obtained as a scaling limit of

[5] To see the relation between (3.17) and the coordinate transformation given in [23], note that the Schwarzian derivative $\{\tan t, t\} = 2$ is constant.

AdS in global coordinates). Insight into this equivalence is provided by the observation that the coordinate transformation (3.19) is such that

$$-2dtd\xi + d\vec{x}^2 = \left(\cos^2\omega T\right)^{-1}\left(-2dTdV - \omega^2\vec{X}^2dT^2 + d\vec{X}^2\right) \,, \tag{3.33}$$

thus exhibiting the conformal flatness of the isotropic plane wave metric appearing on the right hand side. Translated into plane wave lightcone Hamiltonians, it thus conformally relates free particle (untrapped) and isotropic harmonic oscillator (trapped) dynamics. Lifting this transformation to the AdS Poincaré-patch by adding the r-transformation in (3.19) then provides a direct means of establishing (3.32), while the reasoning of section 3.1 provides the additional insight that this coordinate system diagonalises the action of the modified lightcone Hamiltonian $P_+ + \omega^2 K_-$ and the lightcone momentum P_-.

The metric (3.32) captures the $R \to \infty$ (bulk) behaviour of the global $\mathsf{Sch}_{z=2}$ metric (3.20), and, in spite of its similarity to the Poincaré metric, for $\omega > 0$ this is a geodesically complete form of the AdS metric, the trapping harmonic oscillator potential preventing, as above, the geodesics from running off to infinity for finite values of the affine parameter.

Let us conclude this section with some comments on the metric (3.31) for $z > 2$. Recall from section 2 that also for $z > 2$ the Poincaré metric is incomplete but non-singular as $r \to \infty$. It remains to be seen if (3.31) provides a geodesically complete form of the metric also in this case. A new (and perhaps at first disturbing) feature of (3.31) for $z \neq 2$ is its T-dependence. However this is an unavoidable feature of global coordinates for $z > 2$. Indeed, in section 3.1 we established the result that the Schrödinger space-times have no global timelike Killing vector fields. Conversely this implies, just as in the case of de Sitter space, that the metric in global coordinates will necessarily be time-dependent. While this argument does not prove that (3.31) provides a geodesically complete coordinate system for the $z > 2$ metrics, it shows that the time-dependence of (3.31) is no reason to dismiss it. It may in any case be worth understanding if the dependence of the metric on global time is reflected in some manner in non-relativistic scale-invariant field theories with $z > 2$ Schrödinger symmetry.

4 Scalar fields in global coordinates

In order to further study the effect of the trapping term $\omega^2(\vec{X}^2 + R^2)$ of the global metric (3.20), we will look at scalar fields in this section and compare with what is know about scalars fields in the Poincaré patch [1, 2], as well as with the analysis of [5].

Thus, consider the Klein-Gordon equation for a massive complex scalar field ϕ of mass m_0,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) - m_0^2\Phi = 0\,, \tag{4.1}$$

in the global $\mathsf{Sch}_{z=2}$ metric (3.20). We will consider modes $\Phi_{E,m}$ with a definite (trapping) energy $E > 0$ and mass (particle number) $m > 0$, i.e. eigenfunctions of $H + a^2C = \partial_T$ and $P_- = \partial_V$ of the form

$$\Phi_{E,m}(R,\vec{X},T,V) = \mathrm{e}^{-iET}\mathrm{e}^{-imV}\phi(R,\vec{X})\,, \tag{4.2}$$

Introducing spherical coordinates $\{\rho, \text{angles}\}$ in the $\vec{X}$-plane, schematically expanding $\phi(R,\vec{X}) = \sum Y_L \varphi_L(\rho)\phi_L(R)$ using spherical harmonics Y_L (eigenfunctions of the Laplacian on S^{d-1} with eigenvalue $-L(L+d-2)$), one finds that the solutions to the separated ρ-equation that are regular at the origin and well-behaved for $\rho \to \infty$ are given by the functions

$$\varphi_{L,n}(\rho) = \mathrm{e}^{-\frac{1}{2}\omega m \rho^2} \rho^L L_n^{L-1+d/2}(\omega m \rho^2) \ , \tag{4.3}$$

where the $L_n^{L-1+d/2}$ are generalised Laguerre polynomials.

The resulting $\phi_{L,n}(R)$-equation

$$\phi''_{L,n} - \frac{d+1}{R}\phi'_{L,n} + \left(2Em - 4m\omega\left(n + \frac{L}{2} + \frac{d}{4}\right) - \omega^2 m^2 R^2 - \frac{m^2 + m_0^2}{R^2}\right)\phi_{L,n} = 0 \tag{4.4}$$

can then be reduced to a standard confluent hypergeometric differential equation

$$uF''(u) + \left(1 + \frac{\Delta_+ - \Delta_-}{2} - u\right)F' - \left(n + \frac{L}{2} + \frac{d}{4} - \frac{E}{2\omega}\right)F = 0 \tag{4.5}$$

with the ansatz

$$\phi_{L,n}(R) = \mathrm{e}^{-\frac{1}{2}u} u^{\Delta_+/2} F(u) \ , \tag{4.6}$$

where $u = \omega m R^2$ and

$$\Delta_\pm = \frac{d+2}{2} \pm \frac{1}{2}\sqrt{(d+2)^2 + 4(m^2 + m_0^2)} \ , \tag{4.7}$$

The leading asymptotic behaviour of the two linearly independent solutions $\phi^\pm$ as $R \to \infty$ is

$$\phi^\pm_{L,n} \sim \mathrm{e}^{\pm\frac{1}{2}\omega m R^2} \ . \tag{4.8}$$

This is the analogue of the behaviour of the Bessel functions I_ν and K_ν encountered in the standard AdS/CFT correspondence and in the $z = 2$ Poincaré-patch analysis of [1, 2].

The leading behavior of the solution near $R = 0$ can also be deduced from the exact solution, but can more readily be read off directly from (4.4) by neglecting the constant terms and, in particular, the trapping term $\omega^2 m^2 R^2$. The behaviour of the solutions,

$$\phi_{L,n} \sim R^{\Delta_\pm} \ , \tag{4.9}$$

thus necessarily becomes identical to that found in [1, 2] in terms of plane wave Fourier modes $\phi_{\vec{k}}$, namely

$$\phi_{\vec{k}} \sim r^{\Delta_\pm} \ . \tag{4.10}$$

We see that, as in the case of geodesics, the harmonic trapping term has little influence on the dynamics near $R = 0$ but strongly modifies the behaviour as $R \to \infty$. In particular, the solution associated with ϕ^- has the characteristic harmonic oscillator fall-off behaviour

$$\Phi^-_{E,m} \sim \mathrm{e}^{-\frac{1}{2}\omega m(R^2 + \vec{X}^2)} \ . \tag{4.11}$$

To compare with the Hamiltonian analysis of [5], we just note that separating out the V-dependence as in (4.2) turns the Klein-Gordon equation into a Schrödinger equation and that after the unitary transformations that eliminates the first-order derivatives from (4.4) and its counterpart for $\varphi_{L,n}(\rho)$ the corresponding Hamiltonian agrees with the radial Hamiltonians written down in [5].

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References

[1] D.T. Son, *Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry*, *Phys. Rev.* **D 78** (2008) 046003 [arXiv:0804.3972] [SPIRES].

[2] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, *Phys. Rev. Lett.* **101** (2008) 061601 [arXiv:0804.4053] [SPIRES].

[3] S. Kachru, X. Liu and M. Mulligan, *Gravity duals of Lifshitz-like fixed points*, *Phys. Rev.* **D 78** (2008) 106005 [arXiv:0808.1725] [SPIRES].

[4] W.D. Goldberger, *AdS/CFT duality for non-relativistic field theory*, *JHEP* **03** (2009) 069 [arXiv:0806.2867] [SPIRES].

[5] J.L.F. Barbon and C.A. Fuertes, *On the spectrum of nonrelativistic AdS/CFT*, *JHEP* **09** (2008) 030 [arXiv:0806.3244] [SPIRES].

[6] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up Galilean holography*, *JHEP* **11** (2008) 080 [arXiv:0807.1099] [SPIRES].

[7] J. Maldacena, D. Martelli and Y. Tachikawa, *Comments on string theory backgrounds with non- relativistic conformal symmetry*, *JHEP* **10** (2008) 072 [arXiv:0807.1100] [SPIRES].

[8] A. Adams, K. Balasubramanian and J. McGreevy, *Hot spacetimes for cold atoms*, *JHEP* **11** (2008) 059 [arXiv:0807.1111] [SPIRES].

[9] C. Duval, M. Hassaine and P.A. Horvathy, *The geometry of Schrödinger symmetry in gravity background/non-relativistic CFT*, *Annals Phys.* **324** (2009) 1158 [arXiv:0809.3128] [SPIRES].

[10] S.A. Hartnoll and K. Yoshida, *Families of IIB duals for nonrelativistic CFTs*, *JHEP* **12** (2008) 071 [arXiv:0810.0298] [SPIRES].

[11] L. Mazzucato, Y. Oz and S. Theisen, *Non-relativistic branes*, *JHEP* **04** (2009) 073 [arXiv:0810.3673] [SPIRES].

[12] A. Akhavan, M. Alishahiha, A. Davody and A. Vahedi, *Non-relativistic CFT and semi-classical strings*, *JHEP* **03** (2009) 053 [arXiv:0811.3067] [SPIRES].

[13] A. Adams, A. Maloney, A. Sinha and S.E. Vazquez, *$1/N$ effects in non-relativistic gauge-gravity duality*, *JHEP* **03** (2009) 097 [arXiv:0812.0166] [SPIRES].

[14] M. Taylor, *Non-relativistic holography*, arXiv:0812.0530 [SPIRES].

[15] M. Alishahiha, R. Fareghbal, A.E. Mosaffa and S. Rouhani, *Asymptotic symmetry of geometries with Schrödinger isometry*, arXiv:0902.3916 [SPIRES].

[16] S. Schäfer-Nameki, M. Yamazaki and K. Yoshida, *Coset construction for duals of non-relativistic CFTs*, arXiv:0903.4245 [SPIRES].

[17] S.A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, arXiv:0903.3246 [SPIRES].

[18] V. Balasubramanian, P. Kraus and A.E. Lawrence, *Bulk vs. boundary dynamics in anti-de Sitter spacetime*, *Phys. Rev.* **D 59** (1999) 046003 [hep-th/9805171] [SPIRES].

[19] V. Balasubramanian, P. Kraus, A.E. Lawrence and S.P. Trivedi, *Holographic probes of anti-de Sitter space-times*, *Phys. Rev.* **D 59** (1999) 104021 [hep-th/9808017] [SPIRES].

[20] D. Marolf, *States and boundary terms: subtleties of Lorentzian AdS/CFT*, *JHEP* **05** (2005) 042 [hep-th/0412032] [SPIRES].

[21] Y. Nishida and D.T. Son, *Nonrelativistic conformal field theories*, *Phys. Rev.* **D 76** (2007) 086004 [arXiv:0706.3746] [SPIRES].

[22] J. Podolsky, *Interpretation of the Siklos solutions as exact gravitational waves in the anti-de Sitter universe*, *Class. Quant. Grav.* **15** (1998) 719 [gr-qc/9801052] [SPIRES].

[23] D. Brecher, A. Chamblin and H.S. Reall, *AdS/CFT in the infinite momentum frame*, *Nucl. Phys.* **B 607** (2001) 155 [hep-th/0012076] [SPIRES].

[24] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions*, Dover, New York U.S.A. (1974).

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